

A comparison of concepts from computable analysis and effective descriptive set theory

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Computable analysis and effective descriptive set theory are both concerned with complete metric spaces, functions between them and subsets thereof in an effective setting. The precise relationship of the various definitions used in the two disciplines has so far been neglected, a situation this paper is meant to remedy.

As the role of the Cauchy completion is relevant for both effective approaches to Polish spaces, we consider the interplay of effectivity and completion in some more detail.

1 Introduction

Both computable analysis (WEIHRAUCH [28, 31]) and effective descriptive set theory (MOSCHOVAKIS [16]) have a notion of computability on (complete, separable) metric spaces as a core concept. Nevertheless, the definitions are *prima facie* different, and the precise relationship has received little attention so far (contrast e.g. the well-established connections between WEIHRAUCH's and POUR-EL & RICHARD's approach [24] to computable analysis).

The lack of exchange between the two approaches becomes even more regrettable in the light of recent developments that draw on both computable analysis and descriptive set theory:

- The study of Weihrauch reducibility often draws on concepts from descriptive set theory via results that identify various classes of measurable functions as lower cones for Weihrauch reducibility [1, 2, 23]. The Weihrauch lattice is used as the setting for a metamathematical investigation of the computable content of mathematical theorems [3, 9, 20].
- In fact, Weihrauch reducibility was introduced partly as an analogue to Wadge reducibility for functions (see the original papers by WEIHRAUCH [29, 30, 31] and subsequent work by HERTLING [11]), and as such, can itself be seen as a subfield of (effective) descriptive set (or rather function) theory.
- The Quasi-Polish spaces [6] introduced by DE BRECHT allow the generalization of many results from descriptive set theory to a much larger class of spaces (e.g. [5, 18]), and admit a very natural characterization in terms of computable analysis as those countably based spaces with a total admissible Baire-space representation.
- Even more so, the suggested *synthetic descriptive set theory* [22] by the third author and DE BRECHT would extend some fundamental results from descriptive set theory even further,

to general represented spaces [21]. This could pave the way to apply some very strong results by KIHARA [14] to the long-outstanding questions regarding generalizations of the Jayne-Rogers theorem [12, 19, 13, 27].

Our goal with the present paper is to facilitate the transfer of results between the two frameworks by pointing out both similarities and differences between definitions. For example, it turns out that the requirements of an effective metric spaces (as used by MOSCHOVAKIS) are strictly stronger than those WEIHRAUCH imposes on a computable metric space – however, this is only true for specific metrics, by moving to an equivalent metric, the stronger requirements can always be satisfied. Hence, effective Polish spaces and computable Polish spaces are the same concept.

Besides the fundamental layer of metric spaces, we shall also consider the computability structure on hyperspaces such as all Σ_2 -measurable subsets of some given Polish spaces. While these spaces do not carry a meaningful topology, they can nevertheless be studied as represented spaces. This was done implicitly in [16], and more explicitly in [1, 23] and [22].

As a digression, we will consider a more abstract view point on the Cauchy completion to illuminate the different approaches to metric spaces.

2 Effective Polish Spaces and Computable Polish Spaces

We begin by contrasting the definitions of the fundamental structure on metric spaces used to derive computability notions; Moschovakis defines a *recursively presented metric space* (RPMS) and Weihrauch a *computable metric space* (CMS). Throughout the text, by $\nu_{\mathbb{Q}} : \mathbb{N} \rightarrow \mathbb{Q}$ we denote some standard bijection.

Definition(Moschovakis [10]) 1 (1A). Suppose \mathcal{X} is a separable, complete metric space with distance function d . A recursive presentation of \mathcal{X} is any function $\mathbf{r} : \mathbb{N} \rightarrow \mathcal{X}$ whose image $\mathbf{r}[\mathbb{N}] = r_0, r_1, \dots$ is dense in \mathcal{X} and such that the relations

$$P^{d,\mathbf{r}}(i, j, k) \iff d(r_i, r_j) \leq \nu_{\mathbb{Q}}(k)$$

$$Q^{d,\mathbf{r}}(i, j, k) \iff d(r_i, r_j) < \nu_{\mathbb{Q}}(k)$$

are recursive.

Definition(Weihrauch [31]) 1 (cf 8.1.2). We define a computable metric space with its Cauchy representation such that:

1. A computable metric space is a tuple $\mathbf{M} = (M, d, (a_n)_{n \in \mathbb{N}})$ such that (M, d) is a metric space and $(a_n)_{n \in \mathbb{N}}$ is a dense sequence in (M, d) .
2. The Cauchy representation $\delta_{\mathbf{M}} : \mathbb{N}^{\mathbb{N}} \rightarrow M$ is associated with the computable metric space $\mathbf{M} = (M, d, A, \alpha)$ is defined by

$$\delta_{\mathbf{M}}(p) = x : \iff \begin{cases} d(a_{p(i)}, a_{p(k)}) \leq 2^{-i} \text{ for } i < k \\ \text{and } x = \lim_{i \rightarrow \infty} a_{p(i)} \end{cases}$$

3. Finally the following relation

$$\{(t, u, v, w) \mid \nu_{\mathbb{Q}}(t) < d(a_u, a_v) < \nu_{\mathbb{Q}}(w)\} \text{ is r.e.}$$

Both definitions can only ever apply to separable metric spaces, however, a noticeable difference is Moschovakis' requirement of completeness, which is not demanded by Weihrauch. This is only a superfluous distinction, though:

Observation 1. If $\mathbf{M} = (M, d, (a_n)_{n \in \mathbb{N}})$ is a computable metric space (CMS) with a Cauchy representation then its completion $\overline{\mathbf{M}} = (\overline{M}, \overline{d}, (a_n)_{n \in \mathbb{N}})$ (where \overline{d} is the expanded distance function for the completion, specifically $\overline{d}|_M = d$) is also a CMS.

A more substantial difference lies in the decidability-requirement of distances between basic points and rational numbers. For Weihrauch's definition, being able to semi-decide $q < d(a_u, a_w)$ and $d(a_u, a_w) < q$ is enough, whereas Moschovakis demands these to be decidable. By identifying¹ $(a_n)_{n \in \mathbb{N}}$ and $r : \mathbb{N} \rightarrow \mathcal{X}$, we immediately find:

Observation 2. Every recursively presented metric space can be regarded as a computable metric space.

The converse fails in general:

Example 3. Consider the following CMS: Let the base set be $X = \mathbb{N} \uplus \mathbb{N}$, the dense set also X (with a standard bijection) and the distance function be defined as follows (assuming n_i is the i th element of the first copy of \mathbb{N} , n'_i from the second):

$$\begin{aligned} d(n_i, n_j) &:= |n_i - n_j| \\ d(n_i, n'_i) &:= 1 + \frac{1}{s_i} \end{aligned}$$

Where s_i is the step count of the i th Turing machine started with no arguments if it halts

$$d(n_i, n'_i) := 1$$

if it does not. Then, to ensure the validity of the triangle inequality, we set

$$\begin{aligned} d(n_j, n'_i) &:= d(n_i, n'_i) + d(n_i, n_j) \\ d(n'_j, n'_i) &:= d(n_j, n'_i) + d(n'_j, n_j) \end{aligned}$$

This space is a CMS but not an RPMS.

Proof. To output the upper bound $d(n_i, n'_i) < 1 + \frac{1}{k} \leq q_i$ one only has to simulate φ_i , the i th program for k steps, if it did not halt yet, output q_i , if it did halt it will be a lower bound. We can avoid outputting the exact term for the exact step count in case it halts. Similarly we semidecide the other types of distances.

This will form a CMS (with the representation of eventually constant sequences of points).

Suppose towards a contradiction that (X, d) admits a recursive presentation $\mathbf{r} : \mathbb{N} \rightarrow X$. Since the set $\mathbf{r}[\mathbb{N}]$ is dense in (X, d) and the latter space is discrete we have that \mathbf{r} is surjective. It follows easily that there exists a recursive function $f : \mathbb{N} \times \{0, 1\} \rightarrow \mathbb{N}$ such that $\mathbf{r}(f(i, 0)) = n_i$ and $\mathbf{r}(f(i, 1)) = n'_i$.

The decidability-requirements now imply in particular that $d(\mathbf{r}(f(i, 0)), \mathbf{r}(f(i, 1))) = 1$ is a decidable property – but by constructing of d , this would mean that the Halting problem is decidable, providing the desired contradiction. \square

¹That this identification actually makes sense follows from the investigation of the class of computable functions between spaces in Section 3.

We will proceed to find a weaker counterpart to Observation 2. First, note that in a recursively presented metric space we can decide whether $r_n = r_m$?, whereas we cannot decide $a_w = a_u$? in a computable metric space. It is possible, however, to avoid having duplicate points in the dense sequence even in the latter case. First we shall provide a general criterion for when two dense sequences give rise to homeomorphic computable metric spaces:

Lemma 4. For two computable metric spaces $X = (M, d, (a_i)_{i \in \mathbb{N}})$, $X' = (M, d, (a'_i)_{i \in \mathbb{N}})$ if a_i is uniformly computable in X' then the

$$id : X \rightarrow X'$$

identity function is computable.

Proof. The assumption means that there exists a machine N implementing $f : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow (a'_i)_{i \in \mathbb{N}})$ where $f \ n \ k$ is some $a \in \{a'_i\}$ such that $|a_n - a| < 2^{-k}$

Define $id \equiv \lambda x. \lambda n. f \ (x \ (n+1)) \ (n+1)$. By definition $f \ (x \ (n+1)) \ (n+1)$ is a number $a \in \{a'_i\}$ such that $|a_{n+1} - a| < 2^{-n-1}$ where $a_{x \ (n+1)} \in \{a_i\}$ with $|x - a_{x \ (n+1)}| < 2^{-n-1}$, by the triangle inequality $|x - (f \ (x \ (n+1)) \ (n+1))| < 2 \times 2^{-n-1} = 2^{-n}$, therefore it is a valid name of x in X' . \square

In general, we shall write $\mathbf{X} \cong \mathbf{Y}$ iff there is a bijection $\lambda : \mathbf{X} \rightarrow \mathbf{Y}$ such that λ and λ^{-1} are computable. In this paper, λ will generally be the identity on the underlying sets.

Corollary 5. For two computable metric spaces $X = (M, d, (a_i)_{i \in \mathbb{N}})$, $X' = (M, d, (a'_i)_{i \in \mathbb{N}})$ if a_i is uniformly computable in X' and a'_i is uniformly computable in X then $X \cong X'$

As a slight detour, we will prove a more general, but ultimately too weak result. We recall from [21] that a represented space is called computably Hausdorff, if inequality is recognizable. Note that every computable metric space is computably Hausdorff. We define a multivalued map $\text{RemoveDuplicates} : \subseteq \mathcal{C}(\mathbb{N}, \mathbf{X}) \rightrightarrows \mathcal{C}(\mathbb{N}, \mathbf{X})$ by $\text{dom}(\text{RemoveDuplicates}) = \{(x_n)_{n \in \mathbb{N}} \mid \omega = |\{x_n \mid x_n \in \mathbb{N}\}|\}$ and $(y_n)_{n \in \mathbb{N}} \in \text{RemoveDuplicates}((x_n)_{n \in \mathbb{N}})$ iff $\{y_n \mid n \in \mathbb{N}\} = \{x_n \mid n \in \mathbb{N}\}$ and $\forall n \neq m \in \mathbb{N}. y_n \neq y_m$. In words, RemoveDuplicates takes a sequence with infinite range, and produces a sequence with the same range but without duplicates.

Proposition 6. Let \mathbf{X} be computably Hausdorff. Then $\text{RemoveDuplicates} : \subseteq \mathcal{C}(\mathbb{N}, \mathbf{X}) \rightrightarrows \mathcal{C}(\mathbb{N}, \mathbf{X})$ is computable.

Proof. Given a sequence $(x_n)_{n \in \mathbb{N}}$ in a computable Hausdorff space, we can compute $\{n \in \mathbb{N} \mid \forall i < n \ x_i \neq x_n\} \in \mathcal{O}(\mathbb{N})$, i.e. as a recursively enumerable set (relative to the sequence). By assumption on the range of the sequence, this set is infinite. It is a basic result from recursion theory that any recursively enumerable set is the range of an injective computable function, and this holds uniformly. Let λ be such a function. Then $y_n = x_{\lambda(n)}$ satisfies the criteria for the output. \square

The combination of Lemma 4 and Proposition 6 allows us to conclude that for any infinite computable metric space \mathbf{X} , there is a computable metric space \mathbf{X}' with the same underlying set and metric, and a repetition-free dense sequence such that $id : \mathbf{X}' \rightarrow \mathbf{X}$ is computable – but we cannot guarantee computability of $id : \mathbf{X} \rightarrow \mathbf{X}'$ thus. Consequently, we shall employ a more complicated construction:

Theorem 7. For any infinite CMS $\mathbf{X} = (M, d, (a_i)_{i \in \mathbb{N}})$, there is a repetition-free sequence $(a'_i)_{i \in \mathbb{N}}$ such that $\mathbf{X}' = (M, d, (a'_i)_{i \in \mathbb{N}})$ is a CMS with $\mathbf{X} \cong \mathbf{X}'$.

Proof. We will first describe an algorithm obtaining the sequence $(a'_i)_{i \in \mathbb{N}}$ from the original sequence $(a_i)_{i \in \mathbb{N}}$.

1. At any stage, let A' be the finite prefix of the sequence $(a'_i)_{i \in \mathbb{N}}$ fixed so far. We also keep track of a precision parameter n , starting with $n := 1$.
2. In the first stage, we put a_0 into A' (i.e. we set $a'_0 := a_0$)
3. Do the following iteration:
 - (a) Take the next element from $(a_i)_{i \in \mathbb{N}}$ and place it in an auxiliary set B , increment n
 - (b) For all elements $b \in B$, we can compute the number $\min_b \{d(a, b) \mid a \in A'\} \in \mathbb{R}$.
 - (c) For each \min_b , check (non-deterministically) in parallel: if $\min_b < 2^{-n}$ skip b , if $\min_b > 2^{-n-1}$ emit b , remove b from B , add b to A' , repeat.
 - (d) If all elements in B were skipped, repeat.

The parallel test in 3(c) is a common trick in computable analysis. The relations by themselves are not decidable, but as at least one of them has to be true, we can wait until we recognize a true proposition. If the number $\min_b \{d(a, b) \mid a \in A'\}$ lies between 2^{-n-1} and 2^{-n} , then the choice is non-deterministic in the high level view of real numbers as inputs. If all codings and implementations are fixed, then the choice here is determined, too, though.

First, we shall argue that $(a'_i)_{i \in \mathbb{N}}$ is dense and repetition free. If $(a'_i)_{i \in \mathbb{N}}$ were not dense, then there would be some $m, k \in \mathbb{N}$ such that $\forall i \in \mathbb{N} \ d(a_m, a'_i) > 2^{-k}$. However, then once m has been placed into B and n incremented beyond $k + 1$, m would have been chosen for A' – contradiction. The sequence $(a'_i)_{i \in \mathbb{N}}$ cannot have repetitions, because a duplicate element could never satisfy the test in 3(c).

It remains to show that $(a'_i)_{i \in \mathbb{N}}$ is computable in $(a_i)_{i \in \mathbb{N}}$ and vice versa. From Corollary 5 we would then know that $(M, d, (a_i)_{i \in \mathbb{N}}) \cong (M, d, (a'_i)_{i \in \mathbb{N}})$.

By construction $(a'_i)_{i \in \mathbb{N}}$ is computable in (M, d, A) : Given $i \in \mathbb{N}$, just follow the construction above in order to identify which a_j is the i -th element to be put into A' , then we have $a'_i = a_j$.

Now to prove that $(a_i)_{i \in \mathbb{N}}$ is computable in $(M, d, (a'_i)_{i \in \mathbb{N}})$; i.e. that given some $i \in \mathbb{N}$ we can compute a sequence $(n_j)_{j \in \mathbb{N}}$ such that $d(a'_{n_j}, a_i) < 2^{-j}$. For this, we inspect the algorithm above beginning from the point when a_i is put into B . If a_i is moved into A' as the k -th element to enter A' , then $d(a'_k, a_i) = 0$, and we can continue the sequence $(n_j)_{j \in \mathbb{N}}$ as the constant sequence k . If a_i is not moved into A' in the j -th round, then this is due to $\min_{a_i} < 2^{-j}$, and there must be some l such that a'_l witnesses this distance, i.e. $d(a_i, a'_l) < 2^{-j}$. Thus, continuing the sequence with $n_j := l$ works. \square

In [10] by G. and MOSCHOVAKIS, it is proved that for every recursively presented metric space (X, d) there exists a recursive real $0 < \alpha < 1$ such that the metric $\alpha \cdot d$ takes values in $\mathbb{R} \setminus \mathbb{Q} \cup \{0\}$. This idea combined with Theorem 7 gives the following result.

Theorem 8. For every CMS $\mathbf{X} = (M, d, \{a_i\})$ there is a CMS $\mathbf{X}' = (M, \alpha d, \{a'_i\})$ with a computable real $\alpha \leq 1$ such that $\mathbf{X} \cong \mathbf{X}'$, and \mathbf{X}' satisfies the criteria for a recursively presented metric space.

Proof. If \mathbf{X} is finite, the result is straight-forward. If \mathbf{X} is infinite, we may assume by Theorem 7 that $\{a_i\}$ is repetition-free. Let the following be computable bijections:

1. $\langle, \rangle_{-\Delta} : (\mathbb{N} \times \mathbb{N} \setminus \{(n, n) \mid n \in \mathbb{N}\}) \rightarrow \mathbb{N}$
2. $\nu_{\mathbb{Q}}^+ : \mathbb{N} \rightarrow \{q \in \mathbb{Q} \mid q > 0\}$
3. $\langle, \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

Then consider the computable sequence defined via $D_{\langle k, \langle i, j \rangle_{-\Delta} \rangle} = \frac{\nu_{\mathbb{Q}}^+(k)}{d(a_i, a_j)}$. We can diagonalize against the $(D_n)_{n \in \mathbb{N}}$ and find some computable real α with $0.5 \leq \alpha \leq 1$. By choice of α , we find $\alpha d(a_i, a_j) \notin \mathbb{Q}$ for $i \neq j$, hence, the problematic case in the requirements for a recursively presented metric space becomes irrelevant. To compute the identity $\text{id} : \mathbf{X} \rightarrow \mathbf{X}'$, one just needs to map a fast Cauchy sequence $(x_i)_{i \in \mathbb{N}}$ to $(x_{i+1})_{i \in \mathbb{N}}$ (as $\alpha \geq 0.5$), the identity in the other direction does require any changes at all. \square

3 The induced computability structures

A comparison of the computability structure induced by recursive presentations and computable metric spaces respectively is more illuminating in the framework of represented spaces. We recall some notions from [21]: A *represented space* is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$. A multi-valued function between represented spaces is a multi-valued function between the underlying sets. For $f : \subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $F : \subseteq \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$, we call F a *realizer* of f (notation $F \vdash f$), iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \text{dom}(f\delta_X)$. A map between represented spaces is called *computable* (continuous), iff it has a computable (continuous) realizer. Similarly, we call a point $x \in \mathbf{X}$ *computable*, iff there is some computable $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta_X(p) = x$. Any computable metric space induces a represented space via its Cauchy representation, and a function between computable metric spaces is called *computable*, iff it is computable between the induced representations. Note that the realizer-induced notion of continuity coincides with ordinary metric continuity by the main theorem of computable analysis [31].

The category of represented spaces is cartesian closed, meaning we have access to a general function space construction as follows: Given two represented spaces \mathbf{X}, \mathbf{Y} we obtain a third represented space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of functions from X to Y by letting $0^n 1 p$ be a $[\delta_X \rightarrow \delta_Y]$ -name for f , if the n -th Turing machine equipped with the oracle p computes a realizer for f . As a consequence of the UTM theorem, $\mathcal{C}(-, -)$ is the exponential in the category of continuous maps between represented spaces, and the evaluation map is even computable.

Still drawing from [21], we consider the Sierpiński space \mathbb{S} , which allows us to formalize semi-decidability. The computable functions $f : \mathbb{N} \rightarrow \mathbb{S}$ are exactly those where $f^{-1}(\{\top\})$ is recursively enumerable (and thus $f^{-1}(\{\perp\})$ co-recursively enumerable). In general, for any represented space \mathbf{X} we obtain two spaces of subsets of \mathbf{X} ; the space of open sets $\mathcal{O}(\mathbf{X})$ by identifying $f \in \mathcal{C}(\mathbf{X}, \mathbb{S})$ with $f^{-1}(\{\top\})$, and the space of closed sets $\mathcal{A}(\mathbf{X})$ by identifying $f \in \mathcal{C}(\mathbf{X}, \mathbb{S})$ with $f^{-1}(\{\perp\})$. In particular, the computable elements of $\mathcal{O}(\mathbb{N})$ are precisely the recursively enumerable sets. A potential direct definition of a representation for $\mathcal{O}(\mathbb{N})$ is found in $\delta_{rng} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{O}(\mathbb{N})$ defined via $n \in \delta_{rng}(p)$ iff $\exists i \in \mathbb{N} p(i) = n + 1$.

We proceed to introduce the notion of an effective countable base. Note that the countably based computable admissible (see [21] or [26]) are exactly the effective topological spaces studied by WEIHRAUCH, and of these, those that admit a total Baire space representation are the Quasi-Polish spaces introduced by DE BRECHT [6].

Definition 9. An effective countable base for \mathbf{X} is a computable sequence $(U_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X}))$ such that the multivalued partial map $\text{Base} : \subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightrightarrows \mathbb{N}$ is computable. Here $\text{dom}(\text{Base}) = \{(x, U) \mid x \in U\}$ and $n \in \text{Base}(x, U)$ iff $x \in U_n \subseteq U$.

Proposition 10. Let $\mathbf{X} = (M, d, \{a_i\})$ be a CMS. Then $B_{\langle i, j \rangle} = \{x \in M \mid d(x, a_i) < 2^{-j}\}$ provides an effective countable base for \mathbf{X} .

Proof. We start to prove that this is a computable sequence. By the definition of \mathcal{O} , it suffices to show that given x, i, j we can recognize $d(x, a_i) < 2^{-j}$. Let $\delta_M(p) = x$, i.e. $\forall k \, d(x, a_{p(k)}) < 2^{-k}$. Now $d(x, a_i) < 2^{-j}$ iff $\exists k \in \mathbb{N} \, d(a_{p(k)}, a_i) < 2^{-j} - 2^{-k}$. By the conditions on a CMS, the property is r.e., and existential quantification over an r.e. property still produces an r.e. property.

Next, we need to argue that Base is computable. Given some $x \in M$ and some open set $U \in \mathcal{O}(\mathbf{X})$ with $x \in U$, we do know by definition of $\mathcal{O}(\mathbf{X})$ that $x \in U$ will be recognized at some finite stage. Moreover, we can simulate the computation until this happens. At this point, only some finite prefix of the δ_M -name p of x has been read, say of length N . But then we must have $x \in \bigcap_{k \leq N} B_{\langle p(k), k \rangle} \subseteq U$. It is easy to verify that we can identify a particular ball inside the intersection still containing x . \square

We now have the ingredients to give a more specific characterization of both $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ and $\mathcal{O}(\mathbf{X})$ for countably based spaces \mathbf{X}, \mathbf{Y} and computably admissible \mathbf{Y} .

Proposition 11. Let \mathbf{X} have an effective countable base $(U_i)_{i \in \mathbb{N}}$ and be computably separable. Then the map $\bigcup : \mathcal{O}(\mathbb{N}) \rightarrow \mathcal{O}(\mathbf{X})$ defined via $\bigcup(S) = \bigcup_{i \in S} U_i$ is computable and has a computable multivalued inverse.

Proof. That the map is computable follows from [21, Proposition 4.2(4), Proposition 3.3(4)]. For the inverse, let $(a_i)_{i \in \mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathbf{X})$ be a computable dense sequence. Fix some computable realizer of Base . Given some $U \in \mathcal{O}(\mathbf{X})$, test for any $n \in \mathbb{N}$ if $a_n \in U$. If this is confirmed, compute $m_n := \text{Base}(a_n, U)$ and list it in $\bigcup^{-1}(U)$.

It remains to argue that $\bigcup \bigcup^{-1}(U) = U$ with the algorithm described above. If $m \in \bigcup^{-1}(U)$, then by construction $U_m \subseteq U$, hence $\bigcup \bigcup^{-1}(U) \subseteq U$. On the other hand, let $x \in U$. The realizer for Base will choose some m_x on input x, U . As this happens after some finite time, there is some a_x so close to x that the realizer works in exactly the same way². This ensures that m_x is listed in $\bigcup^{-1}(U)$, thus $x \in \bigcup \bigcup^{-1}(U)$. \square

As a consequence of the preceding proposition, we see that for countably based spaces \mathbf{X} , we may conceive of open sets being given by enumeration of basic open sets exhausting them. For computable metric spaces in particular, an open set is given by an enumeration of open balls with basic points as centers and radii of the form 2^{-i} (or equivalently, rational radii):

Definition(Weihrauch [31]) 2 (cf 4.1.2). Given a computable metric space \mathbf{X} , we define a notation ι for the open balls with basic centers and radii of the form 2^{-i} via $\iota(\langle w, 0^k \rangle) = B(\alpha(w), 2^{-k})$.

²For this, it is important to fix one realizer of Base and to use the same name of U for all calls.

Definition(Weihrauch [31]) 3 (5.1.15.4). Define the representation $\theta_{<}^{en} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{O}(\mathbf{X})$ by

$$\theta_{<}^{en}(p) := \bigcup \{I(w) \mid \iota(w) \triangleleft p\}$$

Which is intuitively a name consisting of the descriptions of not necessarily all open balls that exhaust the particular set.

Compare this to:

Definition(Moschovakis [10]) 2 (1B.1). A pointset $G \subseteq \mathcal{X}$ is semirecursive (r.e. open) (Σ_1^0) if

$$G = \bigcup_n N_{\varepsilon(n)} = \bigcup_n N(\mathcal{X}, \varepsilon(n))$$

with some recursive $\varepsilon : \omega \rightarrow \omega$

Proposition 12. For two represented spaces \mathbf{X}, \mathbf{Y} the map $f \mapsto \{(x, U) \mid f(x) \in U\} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \rightarrow \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y}))$ is computable. If \mathbf{Y} is computably admissible, then this map admits a computable inverse.

Proof. That $f \mapsto \{(x, U) \mid f(x) \in U\}$ is computable follows by combining computability of $f \mapsto f^{-1} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \rightarrow \mathcal{C}(\mathcal{O}(\mathbf{Y}), \mathcal{O}(\mathbf{X}))$ and computability of $\in : \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$ and using type conversion.

For the inverse direction, recall that for computably admissible \mathbf{Y} the map $\{U \in \mathcal{O}(\mathbf{Y}) \mid y \in U\} \rightarrow \mathbf{Y} : \subseteq \mathcal{O}(\mathcal{O}(\mathbf{Y}))$ is computable. By computability of $\text{Cut} : \mathbf{X} \times \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y})) \rightarrow \mathcal{O}(\mathcal{O}(\mathbf{Y}))$ defined via $\text{Cut}(x, V) = \{U \mid (x, U) \in V\}$ and composition, we find that $(x_0, \{(x, U) \mid f(x) \in U\}) \mapsto f(x_0) : \subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y})) \rightarrow \mathbf{Y}$ is computable. Currying produces the claim. \square

Corollary 13. Let \mathbf{Y} be computably admissible. Then $f : \mathbf{X} \rightarrow \mathbf{Y}$ is computable iff $\{(x, U) \mid f(x) \in U\} \in \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y}))$ is computable.

This characterization in turn is applicable in effective descriptive set theory:

Definition(Moschovakis [10]) 3 (1D.1 (Dellacherie)). A function $f : X \rightarrow Y$ is recursive if and only if the neighbourhood diagram

$$G^f(x, s) \iff f(x) \in N(Y, s)$$

of f is semirecursive.

4 On Cauchy-completions

An effective version of Cauchy-completion underlies both the definition of computable metric spaces and recursively presented metric spaces. A crucial distinction, though classically vacuous, lies in the question whether spaces embed into their Cauchy-completion. Our goal in this section is to explore the variations upon effective Cauchy-completion, and to subsequently understand the origin of the discrepancy exhibited in Section 2.

Given a represented space \mathbf{X} and some metric d on X , we define the space $\mathcal{S}_C^d(\mathbf{X}) \subseteq \mathcal{C}(\mathbb{N}, \mathbf{X})$ of fast Cauchy sequences by $(x_n)_{n \in \mathbb{N}} \in \mathcal{S}_C^d(\mathbf{X})$ iff $\forall i, j \geq N \ d(x_i, x_j) < 2^{-N}$. If \mathbf{X} is complete, the map $\lim_C : \mathcal{S}_C^d(\mathbf{X}) \rightarrow \mathbf{X}$ is of natural interest (if \mathbf{X} is not complete, we can still study \lim_C as a partial map). In fact, it can characterize admissibility as follows:

Proposition 14. Let \mathbf{X} be computably separable. Let $d : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ be a computable metric, and let $\lim_C : \subseteq \mathcal{C}_S^d(\mathbf{X}) \rightarrow \mathbf{X}$ be computable. Then \mathbf{X} is computably admissible.

Proof. To show that \mathbf{X} is computably admissible, we need to show that $\{x\} \mapsto x : \subseteq \mathcal{K}(\mathbf{X}) \rightarrow \mathbf{X}$ is computable. By definition of compactness and separability, we can find some point a_1 such that $\{x\} \subseteq B(a_1, 2^{-2})$. Then we search for a_2 with $\{x\} \subseteq B(a_2, 2^{-3})$ etc. In these points, we have a fast Cauchy sequence converging to x . \square

Proposition 15. Let \mathbf{X} be computably separable and computably admissible. Let $d : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$ be a computable metric, and let $\{B(a_i, 2^{-n}) \mid i, n \in \mathbb{N}\}$ be an effective countable basis for \mathbf{X} . Then \lim_C^d is computable.

Proof. Given some fast Cauchy sequence $(a_i)_{i \in \mathbb{N}}$, we compute the open balls $B(a_i, 2^{-i})$. By virtue of the open balls forming a basis, we see that $\lim_{i \rightarrow \infty} a_i \in U$ for some open $U \in \mathcal{O}(\mathbf{X})$ iff $\exists n \ B(a_n, 2^{-n}) \subseteq U$, and the latter is actually recognizable. Thus, we can compute $\{U \in \mathcal{O}(\mathbf{X}) \mid \lim_{i \rightarrow \infty} a_i \in U\} \in \mathcal{O}(\mathcal{O}(\mathbf{X}))$. Using the computable admissibility of \mathbf{X} , we can extract $\lim_{i \rightarrow \infty} a_i$ from this set. \square

This characterization of computable metric spaces in terms of fast Cauchy limits of course presupposes the represented space \mathbb{R} with its canonical structure. In particular in the beginnings of computable analysis, various non-standard representations of \mathbb{R} have been investigated. We will investigate what happens to Cauchy completions, if some other represented space \mathbf{R} is used in place of \mathbb{R} .

Definition 16. Let \mathbf{X} be a represented space, such that the metric $d : \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{R}$ is computable. We obtain its Cauchy-closure $\overline{\mathbf{X}}^{d, \mathbf{R}}$ by taking the usual quotient of $\mathcal{S}_C^d(\mathbf{X})$.

Observation 17. Any computable metric space \mathbf{X} embeds³ into its Cauchy-closure $\overline{\mathbf{X}}^{d, \mathbf{R}}$, and d can be extended canonically to $\overline{d} : \overline{\mathbf{X}}^{d, \mathbf{R}} \times \overline{\mathbf{X}}^{d, \mathbf{R}} \rightarrow \mathbf{R}$. A complete computable metric space is defined as the Cauchy-closure of a countable metric space with continuous metric into \mathbb{R} .

Proof. The first part of the claim follows from Proposition 15 in conjunction with Proposition 10. The second part is essentially a reformulation of Observation 1. The third part is immediate from Definition 1. \square

In order to find a contrasting picture of the recursively presented metric spaces, we first introduce the represented space \mathbb{R}_{cf} . Informally, any real number is encoded by its decimal expansion, with infinite repetitions clearly marked⁴. This just ensures that $x \leq q?$ and $x \geq q?$ become both decidable for $x \in \mathbb{R}_{cf}$ and $q \in \mathbb{Q}$.

Observation 18. The space \mathbb{R}_{cf} does not embed into $\overline{\mathbb{R}_{cf}}^{d, \mathbb{R}_{cf}}$. Let $d : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_{cf}$ be a computable metric. In general, $d : \overline{\mathbf{X}}^{d, \mathbb{R}_{cf}} \times \overline{\mathbf{X}}^{d, \mathbb{R}_{cf}} \rightarrow \mathbb{R}_{cf}$ may fail to be computable. A recursively presented metric space is defined as the Cauchy-closure of a countable metric space with continuous metric into \mathbb{R}_{cf} .

Proof. The claims all follow from the observation that $\overline{\mathbb{R}_{cf}}^{d, \mathbb{R}_{cf}} \cong \mathbb{R} \not\cong \mathbb{R}_{cf}$. \square

³A computable embedding $\mathbf{X} \hookrightarrow \mathbf{Y}$ is a computable injection $\iota : \mathbf{X} \rightarrow \mathbf{Y}$ such that the partial inverse ι^{-1} is computable, too.

⁴For example, the unique \mathbb{R}_{cf} -name of $\frac{1}{3}$ is $0.\overline{3}$. The number 1 has the names $0.\overline{9}$ and $1.\overline{0}$.

It is not the case, however, that the space \mathbb{R} would be the only space usable in place of \mathbf{R} when defining the Cauchy-closure to obtain an embedding of a space into its completion. One other example is \mathbb{R}' , the jump⁵ of \mathbb{R} .

5 Representations of point classes

It remains for us to consider the notion of effectivity on higher-order classes of sets (typically called point classes), such as Σ_n^0 -sets ($n > 1$), Borel sets or analytic sets. These have traditionally received little attention in the computable analysis community, with the exception of [1] by BRATTKA. One reason for this presumably was the focus on admissible representations, i.e. spaces carrying a topology – and the natural representations of these classes of sets generally fail to be admissible. The ongoing development of synthetic descriptive set theory does provide representations of all the natural point classes.

All point classes in this section are boldface classes, i.e. we do not consider classes such as effectively Σ_2^0 -sets explicitly. However, as we are concerned with representations (i.e. coding schemes) for the pointclasses, the effective notions can always be recovered by calling a member of some pointclass $\underline{\Gamma}$ *effectively* Γ iff it has a computable name in the respective representation of $\underline{\Gamma}$. As there is no established notion of the right coding scheme up to computable equivalence for generic recursively presented Polish spaces in effective descriptive set theory, our equivalences are only proven up to continuous equivalence. Even in the worst case though any two such coding schemes become computably equivalent relative to some oracle depending only on the pointclass, but not on the individual sets.

The treatment of effectivity on point classes in effective descriptive set theory starts with the notions of universal sets and good universal systems. Let $\underline{\Gamma}$ be a point class, and \mathbf{Z}, \mathbf{X} two spaces⁶. For any $P \subseteq \mathbf{Z} \times \mathbf{X}$ and $z \in \mathbf{Z}$, we write $P_z := \{x \in \mathbf{X} \mid (z, x) \in P\}$. We write $\underline{\Gamma} \upharpoonright \mathbf{X}$ for all the $\underline{\Gamma}$ -subsets of \mathbf{X} . Now we call $G \in \underline{\Gamma} \upharpoonright (\mathbf{Z} \times \mathbf{X})$ a \mathbf{Z} -universal set for $\underline{\Gamma}$ and \mathbf{X} iff $\{P_z \mid z \in \mathbf{Z}\} = \underline{\Gamma} \upharpoonright \mathbf{X}$.

If $\mathbf{Z} = (Z, \delta_Z)$ is a represented space and G a \mathbf{Z} -universal set for $\underline{\Gamma}$ and \mathbf{X} , then we obtain a representation γ_G of $\underline{\Gamma} \upharpoonright \mathbf{X}$ via $\gamma_G(p) = G_{\delta_Z(p)}$. In this situation, we can safely assume that $\mathbf{Z} \subseteq \mathbb{N}^{\mathbb{N}}$, and replace it by $(\text{dom}(\delta_Z), \text{id}_{\text{dom}(\delta_Z)})$ otherwise.

A \mathbf{Z} -universal system for $\underline{\Gamma}$ is an assignment $(G^{\mathbf{X}})_{\mathbf{X}}$ of a \mathbf{Z} -universal set for $\underline{\Gamma}$ and \mathbf{X} for each Polish space \mathbf{X} . If $\mathbf{Z} = \mathbb{N}^{\mathbb{N}}$, we suppress the explicit reference to \mathbf{Z} . A universal system $(G^{\mathbf{X}})_{\mathbf{X}}$ is *good*, if for any for space \mathbf{Y} of the form $\mathbf{Y} = \mathbb{N}^l \times (\mathbb{N}^{\mathbb{N}})^k$ with $l, k \geq 0$ and any Polish space \mathbf{X} there is a continuous function $S^{\mathbf{Y}, \mathbf{X}} : \mathbb{N}^{\mathbb{N}} \times \mathbf{Y} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $(z, y, x) \in G^{\mathbf{Y} \times \mathbf{X}} \Leftrightarrow (S(z, y), x) \in G^{\mathbf{X}}$.

Observation 19. Let γ_H be a representation of $\underline{\Gamma} \upharpoonright \mathbf{X}$ obtained from the universal set H . Further let $(G^{\mathbf{X}})_{\mathbf{X}}$ be a good universal system, and let γ_G be the induced representation of $\underline{\Gamma} \upharpoonright \mathbf{X}$. Then $\text{id} : (\underline{\Gamma} \upharpoonright \mathbf{X}, \gamma_H) \rightarrow (\underline{\Gamma} \upharpoonright \mathbf{X}, \gamma_G)$ is continuous.

Proof. By assumption, $H \in \underline{\Gamma} \upharpoonright (\mathbb{N}^{\mathbb{N}} \times \mathbf{X})$. Hence, there is some $h \in \mathbb{N}^{\mathbb{N}}$ such that $G_h^{\mathbb{N}^{\mathbb{N}} \times \mathbf{X}} = H$. Now $p \mapsto S^{\mathbb{N}^{\mathbb{N}}, \mathbf{X}}(q, p)$ is a continuous realizer of id . \square

⁵The jump of a represented space is discussed in [32, 4, 22].

⁶Usually the spaces involved would be restricted to Polish spaces. However, the formalism is useful for us in a more general setting.

As such, we see that the representations obtained from good universal system for some fixed point class are the weakest one (w.r.t. continuous reducibilities) among those obtained from universal systems in general. Consequently the particular choice of a good universal system can only ever matter for computability considerations, but not for continuity.

We can now contrast the approach to representations of point classes via good universal system with the approach via function spaces and Sierpiński -like spaces underlying [22]. A Sierpiński -like space is a represented space \mathbf{S} with underlying set $\{\top, \perp\}$ - no assumptions on the representation are made. Any such space \mathbf{S} induces a pointclass \mathcal{S} over the represented spaces via $U \in \mathcal{S} \upharpoonright \mathbf{X}$ iff $\chi_U : \mathbf{X} \rightarrow \mathbf{S}$ is continuous⁷ (computable), where $\chi_U(x) = \top$ iff $x \in U$. Note that this approach simultaneously provides for a light-face and a bold-face version of \mathcal{S} . This pointclass comes with a represented space $\mathcal{S}(\mathbf{X})$ via the function space constructor $\mathcal{C}(-, -)$ and identification of a set and its characteristic function.

By the properties of the function space construction, we see that $\ni : \mathcal{S}(\mathbf{X}) \times \mathbf{X} \rightarrow \mathbf{S}$ is computable, which immediately implies that we may interpret \ni as a \mathcal{S} -subset of $\mathcal{S}(\mathbf{X}) \times \mathbf{X}$. Thus, any representation of a fixed slice $\mathcal{S} \upharpoonright \mathbf{X}$ arises from some $\mathcal{S}(\mathbf{X})$ -universal set. By moving along the representation, we may replace $\mathcal{S}(\mathbf{X})$ with some suitable $\mathbf{Z} \subseteq \mathbb{N}^{\mathbb{N}}$ here.

Next, we may relax the requirements for \mathbf{Z} -universal systems for $\tilde{\Gamma}$ to allow \mathbf{Z} to vary as $\mathbf{Z}_{\mathbf{X}}$ with the space \mathbf{X} , and will also let \mathbf{X} range over all represented spaces, rather than just Polish spaces. The resulting notion shall be called a generalized universal system. Such a system $(\mathbf{Z}_{\mathbf{X}}, G^{\mathbf{X}})_{\mathbf{X}}$ is good, if for any represented spaces \mathbf{Y}, \mathbf{X} there is a continuous function $S^{\mathbf{Y}, \mathbf{X}} : \mathbf{Z}_{\mathbf{X} \times \mathbf{Y}} \times \mathbf{Y} \rightarrow \mathbf{Z}_{\mathbf{X}}$ such that $(z, y, x) \in G^{\mathbf{Y} \times \mathbf{X}} \Leftrightarrow (S(z, y), x) \in G^{\mathbf{X}}$.

Observation 20. Let the generalized universal system $(\mathbf{Z}_{\mathbf{X}}, G^{\mathbf{X}})_{\mathbf{X}}$ be obtained from the Sierpiński -like space \mathbf{S} . Then it is good.

Proof. $S^{\mathbf{Y}, \mathbf{X}} : \mathcal{C}(\mathbf{X} \times \mathbf{Y}, \mathbf{S}) \times \mathbf{Y} \rightarrow \mathcal{C}(\mathbf{X}, \mathbf{S})$ is realized via partial function application. \square

For most natural choices of a Sierpiński -like space \mathbf{S} , we may actually replace the occurrence of $\mathcal{C}(\mathbf{X}, \mathbf{S})$ in the induced generalized universal system by $\mathbb{N}^{\mathbb{N}}$ again, thus closing the distance between the two approaches. We recall from [15] that a representation $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbf{X}$ is called precomplete, if for any computable partial $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ there is a computable total $\overline{F} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $\delta \circ F(p) = \delta \circ \overline{F}(p)$ for all $p \in \text{dom}(\delta \circ F(p))$. Now note that if \mathbf{S} admits a precomplete representation, then $\mathcal{C}(\mathbf{X}, \mathbf{S})$ admits a total representation for any \mathbf{X} . Subsequently, we note:

Observation 21. Let the Sierpiński -like space \mathbf{S} admit a precomplete representation. Then it induces a pointclass \mathcal{S} together with a good universal system.

It remains for us to explore which pointclasses on Polish spaces are obtainable from some Sierpiński -like space. First, note that any such class \mathcal{S} is closed under taking preimages under continuous functions. Then, for any Polish space \mathbf{X} and total representation $\delta : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbf{X}$, we observe that $A \in \mathcal{S} \upharpoonright \mathbf{X}$ iff $\delta^{-1}(A) \in \mathcal{S} \upharpoonright \mathbb{N}^{\mathbb{N}}$. Generally, we shall call any pointclass satisfying this property for all total admissible representations of Polish spaces to be $\mathbb{N}^{\mathbb{N}}$ -determined.

Proposition 22. Let $\tilde{\Gamma}$ be $\mathbb{N}^{\mathbb{N}}$ -determined, closed under continuous preimages and admit a good universal system. Then there is some Sierpiński -like space \mathbf{S} with $\tilde{\Gamma} = \mathcal{S}$.

⁷This is continuity in the sense of represented spaces, generally not continuity in a topological setting.

Proof. Let $G \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a universal set for $\mathbb{N}^{\mathbb{N}}$. We define a representation $\delta_G : \mathbb{N}^{\mathbb{N}} \rightarrow \{\top, \perp\}$ via $\delta_G(\langle p, q \rangle) = \top$ iff $(p, q) \in G$. Let the resulting space be \mathbf{S} . We claim that the pointclass induced by \mathbf{S} coincides with $\mathbf{\Gamma}$.

Let $A \in \mathbf{\Gamma} \restriction \mathbf{X}$. Then $\delta_{\mathbf{X}}^{-1}(A) \in \mathbf{\Gamma} \restriction \mathbb{N}^{\mathbb{N}}$. Thus, there is some $a \in \mathbb{N}^{\mathbb{N}}$ with $q \in A \Leftrightarrow (a, q) \in G$. Now $q \mapsto \langle a, q \rangle$ is a continuous realizer of $\chi_A : \mathbf{X} \rightarrow \mathbf{S}$.

Conversely, assume $\chi_A \in \mathcal{C}(\mathbf{X}, \mathbf{S})$. Let $c_A : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ be a continuous realizer of χ_A . Note $((\pi_1, \pi_2) \circ c_A)^{-1}(G) = \delta_{\mathbf{X}}^{-1}(A)$. The left hand side of this equation shows that the set is in $\mathbf{\Gamma}$, as $\mathbf{\Gamma}$ is closed under continuous preimages. The right hand side then implies that $A \in \mathbf{\Gamma} \restriction \mathbf{X}$, as $\mathbf{\Gamma}$ is $\mathbb{N}^{\mathbb{N}}$ -determined. \square

Proposition 23. Let $\mathbf{\Gamma}$ be a pointclass.

1. If $\mathbf{\Gamma}$ is $\mathbb{N}^{\mathbb{N}}$ -determined, then so are $\mathbf{\Gamma}^C$ and $\mathbf{\Gamma} \cap \mathbf{\Gamma}^C$.
2. For countable ordinals α , $\mathbf{\Sigma}_{\alpha}^0$ is $\mathbb{N}^{\mathbb{N}}$ -determined.
3. $\mathbf{\Sigma}_1^1$ is $\mathbb{N}^{\mathbb{N}}$ -determined.

Proof. 1. Just observe that $\delta^{-1}(A^C) = (\delta^{-1}(A))^C$.

2. This is a result by SAINT RAYMOND [25] (cf. [7])

3. Let \mathbf{X} be a Polish space and $A \subseteq \mathbf{X}$ be $\mathbf{\Sigma}_1^1$. Since the inverse image of an analytic set is analytic and δ is continuous it follows that $\delta^{-1}[A] \in \mathbf{\Sigma}_1^1 \restriction \mathbb{N}^{\mathbb{N}}$. Conversely using that as a representation, δ is surjective, we have that $A = \delta[\delta^{-1}[A]]$. So if $\delta^{-1}[A]$ is a $\mathbf{\Sigma}_1^1$ subset of $\mathbb{N}^{\mathbb{N}}$ it follows from the closure of $\mathbf{\Sigma}_1^1$ under continuous images that $A \in \mathbf{\Sigma}_1^1 \restriction \mathbf{X}$. \square

Corollary 24. The approaches to continuity and computability for $\mathbf{\Sigma}_{\alpha}^0$ and $\mathbf{\Sigma}_1^1$ from effective descriptive set theory and synthetic descriptive set theory coincide.

A very important pointclass not yet proven to receive equivalent treatment are the Borel sets \mathcal{B} , alternatively $\mathbf{\Delta}_1^1$ by Suslin's theorem (e.g. [17]). There cannot be any $\mathbb{N}^{\mathbb{N}}$ -universal Borel sets⁸ – however, there are \mathbf{B} -universal sets for $\mathbf{\Delta}_1^1$, with non-Polish \mathbf{B} . Such a set can be obtained from the Borel codes used in effective descriptive set theory. We currently cannot prove uniform equivalence of the two approaches for Borel sets on arbitrary Polish spaces, as this would require a uniform version of SAINT RAYMOND's result in [25]. Thus, we first provide a non-uniform treatment of Borel sets on arbitrary Polish spaces, and then a uniform treatment of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

Definition 25. ([16] 3H) The set of *Borel codes* $\mathbf{BC} \subseteq \mathbb{N}^{\mathbb{N}}$ is defined by recursion as follows

$$\begin{aligned} p \in \mathbf{BC}_0 &\iff p(0) = 0 \\ p \in \mathbf{BC}_{\alpha} &\iff p = 1\langle p_0, p_1, \dots \rangle \ \& \ (\forall n)(\exists \beta < \alpha)[p_n \in \mathbf{BC}_{\beta}] \\ \mathbf{BC} &= \cup_{\alpha} \mathbf{BC}_{\alpha} \text{ for all countable ordinals } \alpha. \end{aligned}$$

With an easy induction one can see that $\mathbf{BC}_{\alpha} \subseteq \mathbf{BC}_{\beta}$ for all $\alpha < \beta$ and that \mathbf{BC}_{α} is a Borel set.

⁸Any such set would fall into $\mathbf{\Sigma}_{\alpha}^0$ for some countable ordinal α , but then cannot have any set $A \in \mathbf{\Sigma}_{\alpha+1}^0 \setminus \mathbf{\Sigma}_{\alpha}^0$ as a section.

For all $p \in \text{BC}$ we denote by $|p|$ the least ordinal α such that $p \in \text{BC}_\alpha$. It is not hard to verify that

$$|1\langle p_0, p_1, \dots \rangle| = \sup_{n \in \mathbb{N}} |p_n| + 1$$

Let \mathbf{X} be a Polish space, and $\delta_{\mathcal{O}} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{O}(\mathbf{X})$ a standard representation of its open sets. For all countable ordinals α we define the function $\pi_\alpha^{\mathbf{X}} : \text{BC}_\alpha \rightarrow \mathcal{B} \upharpoonright \mathbf{X}$ recursively by

$$\begin{aligned} \pi_0^{\mathbf{X}}(0p) &= \delta_{\mathcal{O}}(p) \\ \pi_\alpha^{\mathbf{X}}(1\langle p_0, p_1, \dots \rangle) &= \pi_{|p_n|}^{\mathbf{X}}(p_n)^C. \end{aligned}$$

An easy induction shows that the function $\pi_\alpha^{\mathbf{X}}$ is onto $\Sigma_\alpha^0 \upharpoonright \mathbf{X}$, and that $\pi_\beta^{\mathbf{X}} \upharpoonright \text{BC}_\alpha = \pi_\alpha^{\mathbf{X}}$ for all $\alpha < \beta$. So one can define the *Borel coding* $\pi^{\mathbf{X}} : \text{BC} \rightarrow \mathbf{X}$ by

$$\pi^{\mathbf{X}}(p) = \pi_{|p|}^{\mathbf{X}}(p).$$

so that the family $\Sigma_\alpha^0 \upharpoonright \mathbf{X}$ is exactly the family of all $\pi^{\mathbf{X}}(p)$ for $p \in \text{BC}_\alpha$, in particular a set $A \subseteq \mathbf{X}$ is Borel exactly when $A = \pi^{\mathbf{X}}(p)$ for some $p \in \text{BC}$.

Lemma 26. The following are more or less well-known facts in descriptive set theory:

1. For all countable ordinals α the set $\{p \in \text{BC} \mid |p| \leq \alpha\}$ is Borel.

Proof. This is because $\{p \in \text{BC} \mid |p| \leq \alpha\} = \text{BC}_\alpha$. □

2. The set BC is a Π_1^1 subset of $\mathbb{N}^{\mathbb{N}}$ and so in particular it is a $\mathbf{\Pi}_1^1$ set.

Proof. The latter is a consequence of 7C.8 in [16], since one can see that the set BC is the least fixed point of a suitably chosen monotone operation. □

3. There exists a Σ_1^1 relation $\leq_\Sigma \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that for all $p \in \text{BC}$ and all $q \in \mathbb{N}^{\mathbb{N}}$ we have that

$$[q \in \text{BC} \ \& \ |q| \leq |p|] \iff q \leq_\Sigma p.^9$$

Proof. Note that $|1\langle q_0, q_1, \dots \rangle| \leq |1\langle p_0, p_1, \dots \rangle|$ iff $\exists t \in \mathbb{N}^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N} \ |q_n| \leq |p_{t(n)}|$, assuming $q_i, p_i \in \text{BC}$. Building upon this idea, consider the closed relation R defined as the least fixed point of:

$$R(p, q, \langle t', \langle t_0, t_1, \dots \rangle \rangle) :\Leftrightarrow q(0) = 0 \vee (p = 1\langle p_0, p_1, \dots \rangle \wedge q = 1\langle q_0, q_1, \dots \rangle \wedge \forall n \in \mathbb{N} \ R(p_n, q_{t'(n)}, t_n))$$

Now $q \leq_\Sigma p :\Leftrightarrow \exists t \in \mathbb{N}^{\mathbb{N}} \ R(p, q, t)$ is a Σ_1^1 relation, and satisfies our criterion. □

4. The set BC is not a Borel subset of $\mathbb{N}^{\mathbb{N}}$.

⁹The idea behind this condition can be found in the notion of Γ -norms, see [16] 4B. In the same way, one could also obtain a $\mathbf{\Pi}_1^1$ -relation with the same property (note that it does not follow that there is a $\mathbf{\Delta}_1^1$ -relation).

Proof. We will show that if BC were Borel, then the set of well-founded trees would be analytic, which is a contradiction (as shown e.g. in [8, Section 11.8]).

Note that a tree¹⁰ T is well-founded iff there exists an assignment $P : T \rightarrow BC$ such that for all $u, v \in T$ if v extends u then $|P(v)| < |P(u)|$.

This is easy to see: If T is well-founded then we use bar recursion to get P such that $|P(u)| = \sup |P(un)| + 1$. Conversely if P is such an assignment and T contained an infinite branch then we would get a strictly decreasing sequence of ordinals, a contradiction.

Now condition $|P(v)| < |P(u)|$ can be replaced by $S(P(v)) \leq_\Sigma P(u)$, with \leq_Σ as above, and S is a continuous function such that $|S(q)| = |q| + 1$. Thus, we have

$$T \text{ is well-founded} \Leftrightarrow \exists P \quad \forall u, v \in \mathbb{N}^* \quad S(P(v)) \leq_\Sigma P(u)$$

which provides the claimed equivalence. \square

5. Let $f : \mathbb{N}^\mathbb{N} \rightarrow BC$ be Borel measurable. Then there is a countable ordinal α such that $\forall p \in \mathbb{N}^\mathbb{N} \quad |f(p)| \leq \alpha$.

Proof. If this were not the case we would have that

$$q \in BC \iff (\exists p)[q \leq_\Sigma f(p)],$$

where \leq_Σ is as above. Since f is Borel measurable the preceding equivalence would imply that the set BC is a Σ_1^1 subset of $\mathbb{N}^\mathbb{N}$. Hence from the Suslin Theorem it would follow that BC is a Borel set, a contradiction and our claim is proved. \square

Definition 27. We define the Sierpiński-like space $\mathbf{S}_\mathcal{B} = (\{\perp, \top\}, \delta_\mathcal{B})$ recursively via

$$\begin{aligned} \delta_\mathcal{B}(p) \text{ is defined} &\iff p \in BC \\ \delta_\mathcal{B}(0p) &= \delta_\mathcal{B}(p) \\ \delta_\mathcal{B}(1\langle p_0, p_1, \dots \rangle) &= \bigwedge_{i \in \mathbb{N}} \neg \delta_\mathcal{B}(p_i). \end{aligned}$$

Note that by construction of $\mathbf{S}_\mathcal{B}$, we find that $\in : \mathbb{N}^\mathbb{N} \times \mathcal{B} \rightarrow \mathbf{S}_\mathcal{B}$ is computable.

Proposition 28. Fix a Polish space \mathbf{X} . For $A \subseteq \mathbf{X}$ we find the following to be equivalent:

1. $A \in \mathcal{B} \upharpoonright \mathbf{X}$
2. $\chi_A : \mathbf{X} \rightarrow \mathbb{S}_\mathcal{B}$ is continuous.
3. $\chi_A : \mathbf{X} \rightarrow \mathbb{S}_\mathcal{B}$ is Borel measurable.

Proof. 1. \Rightarrow 2. Fix a total admissible representation $\delta_\mathbf{X} : \mathbb{N}^\mathbb{N} \rightarrow \mathbf{X}$. Let us assume that $A \in \mathcal{B} \upharpoonright \mathbf{X}$. Then $\delta_\mathbf{X}^{-1}(A) \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$. If a is a Borel code for $\delta_\mathbf{X}^{-1}(A)$, then $q \mapsto \in(q, a)$ is a continuous realizer for $\chi_A : \mathbf{X} \rightarrow \mathbb{S}_\mathcal{B}$.

2. \Rightarrow 3. Trivial.

¹⁰Here we understand a tree to be a subset of \mathbb{N}^* that is closed under taking prefixes.

3. \Rightarrow 1. Now let us assume that $\chi_A : \mathbf{X} \rightarrow \mathbf{S}_B$ is Borel measurable. Let $c_A : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ be a Borel measurable realizer of χ_A . We remark that $c_A(p) \in \text{BC}$ for all $p \in \mathbb{N}^\mathbb{N}$. Consider now some countable ordinal α_A such that $|c_A(p)| < \alpha_A$ for all $p \in \mathbb{N}^\mathbb{N}$, which we may obtain from Lemma 26 (5). The set $S_{\alpha_A} := \{p \in \mathbb{N}^\mathbb{N} \mid \delta_B(p) = \top \wedge |p| \leq \alpha_A\}$ is a Borel subset of $\mathbb{N}^\mathbb{N}$. Then $c_A^{-1}(S_{\alpha_A}) = \delta_{\mathbf{X}}^{-1}(A)$ is Borel as well and hence it is $\Sigma_{\beta_A}^0$ for some countable ordinal β_A . By Proposition 23 (2), we find that $A \in \Sigma_{\beta_A}^0 \upharpoonright \mathbf{X}$, in particular, A is Borel. \square

As announced above, we will proceed to show that for Baire space the representation of \mathcal{B} via Borel codes is computably equivalent to the representation via the function space into \mathbf{S}_B . In this, we will consider the Borel codes to be the default representation of $\mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$. A new ingredient of the proof will be:

Lemma 29. The operation $r : \subseteq \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N} \rightarrow \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$ with $\text{dom}(r) = \{A \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N} \mid A \subseteq \text{BC}\}$ and $r(A) = \{p \in A \mid \delta_B(p) = \top\}$ is well-defined and computable.

Proof. We start by providing Σ_1^1 -sets T and B , such that $\delta_B(p) = \top \Leftrightarrow p \in \text{BC} \cap T$ and $\delta_B(p) = \perp \Leftrightarrow p \in \text{BC} \cap B$. This is done by constructing two Π_1^0 -sets $P, Q \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ via an interleaving fixed point construction¹¹:

$$\begin{aligned} (0p, q) \in P & \Leftrightarrow p = 0^\mathbb{N} \\ (0p, nq) \in Q & \Leftrightarrow p(n) = 1 \\ (1\langle p_0, p_1, \dots \rangle, \langle q_0, q_1, \dots \rangle) \in P & \Leftrightarrow \forall n \in \mathbb{N} (p_n, q_n) \in Q \\ (1\langle p_0, p_1, \dots \rangle, nq) \in Q & \Leftrightarrow (p_n, q) \in P \end{aligned}$$

Now $p \in T \Leftrightarrow \exists q \in \mathbb{N}^\mathbb{N} (p, q) \in P$ and $p \in B \Leftrightarrow \exists q \in \mathbb{N}^\mathbb{N} (p, q) \in Q$ are our desired sets.

Given $A \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$ we can compute $A \cap T \in \Sigma_1^1 \upharpoonright \mathbb{N}^\mathbb{N}$ and $A^C \cup B \in \Sigma_1^1 \upharpoonright \mathbb{N}^\mathbb{N}$, and note that $A \subseteq \text{BC}$ implies $(A \cap T)^C = A^C \cup B$, so by applying the effective Suslin theorem (MOSCHOVAKIS [17]) we can obtain $r(A) = A \cap T \in \mathcal{B}$. \square

Theorem 30. The map $A \mapsto \chi_A : \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N} \rightarrow \mathcal{C}(\mathbb{N}^\mathbb{N}, \mathbf{S}_B)$ is a computable isomorphism.

Proof. That this map is computable follows by currying from the computability of $\in : \mathbb{N}^\mathbb{N} \times \mathcal{B} \rightarrow \mathbf{S}_B$; that is a bijection from Proposition 28. It only remains to prove that its inverse is computable, too.

Given $\chi_A \in \mathcal{C}(\mathbb{N}^\mathbb{N}, \mathbf{S}_B)$, we can compute the Σ_1^1 set $\chi_A[\mathbb{N}^\mathbb{N}]$. Then we use the effective Suslin theorem (MOSCHOVAKIS [17]) on $\chi_A[\mathbb{N}^\mathbb{N}]$ and the Σ_1^1 -set BC^C to obtain some $B \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$ with $\chi_A[\mathbb{N}^\mathbb{N}] \subseteq B \subsetneq \text{BC}$. Using the computable map r from Lemma 29 we can then obtain $A \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$ as $A = \chi_A^{-1}(r[B])$. \square

¹¹The reader coming from a computable analysis background may prefer to see the following as instructions for a dove-tailing programme trying to disprove $(p, q) \in P$ or $(p, q) \in Q$ by unraveling the instructions. If ever one of the first two cases is reached and yields a negative answer, this is propagated back and disproves the original membership query. It is perfectly fine to have ill-founded computation paths, these can never yield contradictions and thus may cause queries to fall in P or Q where the first parameter is not a Borel code.

6 Conclusions

We have demonstrated that the computability notions used in computable analysis (and synthetic descriptive set theory) and effective descriptive set theory respectively coincide for objects in the scope of both. When it comes to metric spaces, the scope of effective descriptive set theory is more restrictive, however, the difference disappears modulo a rescaling of the metric. While the requirements for pointclasses to be treatable in the two frameworks differ significantly, the computability notions for Σ_α^0 , Π_α^0 , Σ_1^1 and Π_1^1 coincide for Polish spaces, and $\mathcal{B}(\Delta_1^1)$ is the same in both frameworks for Baire space.

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